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The Discrete Legendre Transform of Sequences of Exponential Growth

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The discrete Legendre transformation is investigated in a space of complex-valued sequences of exponential growth and its dual. Inversion and uniqueness theorems are established and some operational properties are obtained. The translation operator and the convolution are also studied. The theory developed is used in solving certain partial finite-difference equations.

1. INTRODUCTION

The finite Legendre transformation

$$\{l^* f(x)\}(n) = F(n) = \int_{-1}^{1} P_n(x) f(x)\, dx, \tag{1.1}$$

where the $P_n(x)$ are the well-known Legendre polynomials [3, 12] has been studied from a classical approach by different authors, among others, by C. J. Tranter [13], R. V. Churchill and G. L. Dolph [1], and I. N. Sneddon [11]. The corresponding inversion formula is given by the Fourier-Legendre series expansion:

$$\{l^{*-1} F(n)\}(x) = f(x) = \sum_{n=0}^{\infty} \frac{2n+1}{2} P_n(x) F(n). \tag{1.2}$$

The analysis of this transform in a space of generalized functions is tackled mainly by A. H. Zemanian [14, 15] but in a more general framework that includes all the finite transformations of Sturm-Liouville type.

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In [8] we introduce the space $\mathscr{L}(-1,1)$ of all functions $\varphi(x) \in C^{\infty}(-1,1)$ such that

$$\gamma_k(\varphi) = \sup_{-1<x<1} |R_x^k \varphi(x)| < \infty,$$

for every $k = 0, 1, 2, \ldots$, R_x being the differential operator

$$R_x = D(x^2 - 1)D = (x^2 - 1)D^2 + 2xD, \qquad D = \frac{d}{dx}.$$

We assign to $\mathscr{L}(-1,1)$ the topology generated by the family of seminorms $\{\gamma_k\}_{k \in \mathbf{N}_0}$. Thus, it can be shown that $\mathscr{L}(-1,1)$ is a Fréchet space.

Remember that [3, (II), p. 179]

$$R_x^k P_n(x) = [n(n+1)]^k P_n(x), \qquad k = 0, 1, 2, \ldots, \tag{1.3}$$

$\mathscr{L}'(-1,1)$ stands for the dual space of $\mathscr{L}(-1,1)$. Considered as a function of x, $P_n(x) \in \mathscr{L}(-1,1)$, so the generalized transform of $f(x) \in \mathscr{L}'(-1,1)$ will be defined by means of

$$(l^{*\prime} f)(n) = F(n) = \langle f(x), P_n(x) \rangle. \tag{1.4}$$

The translation operator associated with the transform (1.1) is

$$T_y \varphi(x) = \frac{1}{\pi} \int_0^{\pi} \varphi\Big(xy + \sqrt{(1-x^2)(1-y^2)} \cos\alpha\Big)\, d\alpha. \tag{1.5}$$

Now, the convolution is introduced as usual through [1, 9, 11]

$$(\varphi * \psi)(x) = \int_{-1}^{1} \varphi(y) T_x \psi(y)\, dy. \tag{1.6}$$

In this paper the situation is inverted in the way that (1.2) will be used to define the direct transform

$$\{lF(n)\}(x) = f(x) = \sum_{n=0}^{\infty} \frac{2n+1}{2} F(n) P_n(x) \tag{1.7}$$

which we call the discrete Legendre transformation, its inversion formula being supplied on the contrary by

$$\{l^{-1} f(x)\}(n) = F(n) = \int_{-1}^{1} P_n(x) f(x)\, dx. \tag{1.8}$$

An interesting classical study of (1.7)–(1.8) can be found in [5, 6]. The main objective of this work is to extend the discrete Legendre transform to a distributional space. For this purpose we establish in Section 2 that $P_n(x)$, considered as a function of n, belongs to a certain testing function space of complex-valued sequences of exponential growth. Once the generalized discrete transform has been defined in Section 3, inversion and uniqueness theorems are proved and an operational calculus is generated. The translation operator and the convolution associated with the discrete transform (1.7) are considered in Section 4 in adequate spaces. Finally, the results obtained are applied to solve some difference equations.

We shall frequently use the recurrence relation [3, (II), p. 179 (9)],

$$(n+1)P_{n+1}(x) + nP_{n-1}(x) = (2n+1)xP_n(x). \tag{1.9}$$

2. THE DISCRETE LEGENDRE TRANSFORMATION IN THE SPACE $l'(\mathbf{N}_0)$

If $\mathbf{N}_0$ denotes the set of nonnegative integer numbers, the space $l(\mathbf{N}_0)$ consists of all the complex-valued sequences $\{\Phi(n)\}_{n\in\mathbf{N}_0}$ such that

$$\lambda_{\alpha,k}(\Phi(n)) = \sup_{n\in\mathbf{N}_0} |e^{-\alpha n}\Lambda_n^{*k}\Phi(n)| < \infty, \tag{2.1}$$

where α is a positive parameter, for every $k = 0, 1, 2, \dots$. Here, Λ_n^* stands for the finite-difference operator

$$\Lambda_n^*\Phi(n) = \frac{n+1}{2n+3}\Phi(n+1) + \frac{n}{2n-1}\Phi(n-1). \tag{2.2}$$

Henceforth $\Phi(n)$ must be taken as 0 if n is negative. The adjoint operator of Λ_n^* is given by

$$\Lambda_n\Phi(n) = \frac{(n+1)\Phi(n+1) + n\Phi(n-1)}{2n+1}. \tag{2.3}$$

When endowed with the topology generated by the collection of seminorms (2.1), $l(\mathbf{N}_0)$ becomes a Fréchet space. The dual space of $l(\mathbf{N}_0)$ is denoted by $l'(\mathbf{N}_0)$.

We now list some properties of these spaces:

(a) The sequence $((2n+1)/2)P_n(x) \in l(\mathbf{N}_0)$ for each x, $-1 \le x \le 1$. Indeed, from (2.2) and (1.9) we infer quickly that

$$\Lambda_n^*\left\{\frac{2n+1}{2}P_n(x)\right\} = x\frac{2n+1}{2}P_n(x).$$

By applying repeatedly this result we arrive at

$$\Lambda_n^{*k}\left\{\frac{2n+1}{2}P_n(x)\right\} = x^k\frac{2n+1}{2}P_n(x), \qquad k \in \mathbf{N}_0. \tag{2.4}$$

This formula plays the same role in relation to the discrete transformation (1.7) that (1.3) does for the Legendre transform (1.1).

Hence,

$$\lambda_{\alpha,k}\left\{\frac{2n+1}{2}P_n(x)\right\} = \sup_{n\in\mathbf{N}_0}\left|e^{-\alpha n}x^k\frac{2n+1}{2}P_n(x)\right| < \infty,$$

since $|x^kP_n(x)| \le 1$, $-1 \le x \le 1$.

(b) If $\{F(n)\}_{n\in\mathbf{N}_0}$ is a complex-valued sequence such that

$$\sum_{n=0}^{\infty} e^{\alpha n}|F(n)| < \infty, \tag{2.5}$$

then $F(n)$ gives rise to a regular member in $l'(\mathbf{N}_0)$ through

$$\langle F(n), \Phi(n)\rangle = \sum_{n=0}^{\infty} F(n)\Phi(n), \qquad \Phi \in l(\mathbf{N}_0). \tag{2.6}$$

In fact, F is clearly linear. On the other hand, we have for every $\Phi(n) \in l(\mathbf{N}_0)$,

$$|\langle F(n), \Phi(n)\rangle| \le \sup_{n\in\mathbf{N}_0}|e^{-\alpha n}\Phi(n)|\sum_{n=0}^{\infty} e^{\alpha n}|F(n)| = C\lambda_{\alpha,0}(\Phi),$$

which shows that (2.6) truly defines a functional F on $l(\mathbf{N}_0)$.

(c) The difference operator Λ_n^*, given by (2.2), is a continuous linear mapping of $l(\mathbf{N}_0)$ into itself, as is immediately implied by

$$\lambda_{\alpha,k}\{\Lambda_n^*\Phi(n)\} = \lambda_{\alpha,k+1}\{\Phi(n)\}, \qquad \Phi \in l(\mathbf{N}_0).$$

Consequently, the generalized difference operator Λ_n defined as the adjoint of the classical operator Λ_n^* by means of

$$\langle \Lambda_n F(n), \Phi(n)\rangle = \langle F(n), \Lambda_n^*\Phi(n)\rangle, \qquad F \in l'(\mathbf{N}_0), \Phi \in l(\mathbf{N}_0), \tag{2.7}$$

is also a continuous linear mapping of $l'(\mathbf{N}_0)$ into itself.

Notice that definition (2.7) is consistent with the usual operational rules of the distributional calculus.

(d) Finally, we note the relation between both of the operators Λ_n and Λ_n^*

$$\Lambda_n \Phi(n) = \frac{2}{2n+1} \Lambda_n^* \frac{2n+1}{2} \Phi(n) \tag{2.8}$$

$$\Lambda_n^* \Phi(n) = \frac{2n+1}{2} \Lambda_n \frac{2}{2n+1} \Phi(n). \tag{2.9}$$

3. THE GENERALIZED DISCRETE LEGENDRE TRANSFORMATION

For an arbitrary generalized sequence $F(n) \in l'(\mathbf{N}_0)$, we define its discrete Legendre transform as the application of $F(n)$ to the kernel-sequence $((2n+1)/2)P_n(x)$; i.e.,

$$\{l'F(n)\}(x) = f(x) = \left\langle F(n), \frac{2n+1}{2} P_n(x) \right\rangle \tag{3.1}$$

for every x, $-1 < x < 1$. This definition makes sense because $((2n+1)/2)P_n(x) \in l(\mathbf{N}_0)$, as was seen in part (a) of the preceding section.

PROPOSITION 3.1. *Let $F(n)$ be any member of $l'(\mathbf{N}_0)$ and denote $f(x)$ as its generalized discrete Legendre transform. The, $f(x)$ is an infinitely differentiable function and one has*

$$D^r f(x) = \left\langle F(n), \frac{2n+1}{2} \frac{d^r}{dx^r} P_n(x) \right\rangle, \tag{3.2}$$

for each $r = 0, 1, 2, \ldots$.

Proof. In view of (2.4) we may write

$$\Lambda_n^{*k} \left[\frac{2n+1}{2} \frac{d^r}{dx^r} P_n(x) \right] = \frac{d^r}{dx^r} \left[x^k \frac{2n+1}{2} P_n(x) \right]$$
$$= \frac{2n+1}{2} \sum_{j=0}^{r} \binom{r}{j} \frac{k!}{(k-r+j)!} x^{k-r+j} P_n^{(j)}(x).$$

By taking into account the last expression and the fact that $|P_n^{(j)}(x)| \le n^{2j}$, $-1 \le x \le 1$ [2, p. 233], it is inferred

$$\begin{aligned}&\left| e^{-\alpha n} \Lambda_n^{*k} \frac{2n+1}{2} \frac{d^r}{dx^r} P_n(x) \right| \\ &\qquad \le e^{-\alpha n} \frac{2n+1}{2} \sum_{j=0}^{r} \binom{r}{j} k(k-1)\cdots(k-r+j+1) n^{2j} \\ &\qquad \le C_{\alpha,r} r! k^r < \infty, \qquad n \in \mathbf{N}_0,\end{aligned}$$

for certain positive constant $C_{\alpha,r}$. Therefore, $((2n+1)/2)(d^r/dx^r)P_n(x) \in l(\mathbf{N}_0)$, $r = 0, 1, 2, \ldots,$ and the right-hand side in (3.2) makes sense. To get the equality (3.2), we start with the case $r = 1$. Then, if $x \in (-1, 1)$ we choose $h \in \mathbb{R}$ such that $x + h \in (-1, 1)$ as well and put

$$\frac{f(x+h) - f(x)}{h} - \left\langle F(n), \frac{2n+1}{2} P_n'(x) \right\rangle = \langle F(n), \Phi(n; x, h) \rangle, \tag{3.3}$$

where

$$\Phi(n; x, h) = \frac{2n+1}{2} \left[\frac{P_n(x+h) - P_n(x)}{h} - P_n'(x) \right].$$

But

$$\begin{aligned}\Lambda_n^{*k} \Phi(n; x, h) = \frac{2n+1}{2} \Bigg[& \frac{(x+1)^h P_n(x+h) - x^k P_n(x)}{h} \\ & - \frac{d}{dx}\big(x^k P_n(x)\big) \Bigg]\end{aligned}$$

in line with (2.5) and the fact that

$$\Lambda_n^* \left\{ \frac{2n+1}{2} D_x P_n(x) \right\} = D_x \Lambda_n^* \left(\frac{2n+1}{2} P_n(x) \right) = \frac{2n+1}{2} D_x\big(x P_n(x)\big).$$

Then, we may write

$$\begin{aligned}\lambda_{\alpha,k}\{\Phi(n; x, h)\} = & \sup_{n \in \mathbf{N}_0} e^{-\alpha n} \\ & \times \frac{2n+1}{2|h|} \left| \int_0^h \int_0^\lambda \frac{d^2}{du^2} \big\{ (x+u)^k P_n(x+u) \big\}\, du\, d\lambda \right|.\end{aligned} \tag{3.4}$$

Moreover, by resorting again to the boundedness of the derivatives of Legendre polynomials [2, p. 223, Theorem 1] we are led to

$$\left|\frac{1}{h}\int_0^h\int_0^\lambda \frac{d^2}{du^2}\left\{(x+u)^k P_n(x+u)\right\} du\, d\lambda\right|$$
$$= \left|\frac{1}{h}\int_0^h\int_0^\lambda \left[(x+u)^k P_n''(x+u) + 2k(x+u)^{k-1}P_n'(x+u)\right.\right.$$
$$\left.\left. + k(k-1)(x+u)^{k-2}P_n(x+u)\right] du\, d\lambda\right|$$
$$\leq \frac{1}{|h|}\left|\int_0^h\int_0^\lambda \left[n^4 + 2kn^2 + k(k-1)\right] du\, d\lambda\right|$$
$$= \left(n^4 + 2kn^2 + k(k-1)\right)\frac{|h|}{2}.$$

By combining this last result and (3.4) we conclude

$$\lambda_{\alpha,k}\{\Phi(n; x, h)\} \leq C(\alpha, k)|h| \to 0, \qquad \text{as } h \to 0,$$

$C(\alpha, k)$ being a suitable positive constant, uniformly on $n \in \mathbf{N}_0$. Consequently, we have established that $\Phi(n; x, h) \to 0$ as $h \to 0$ in the topology of the space $l(\mathbf{N}_0)$. This implies, in view of (3.3), that expression (3.2) is valid for $r = 1$. The general case follows by an inductive argument on r.

PROPOSITION 3.2. *If $F(n)$ is an arbitrary member of $l'(\mathbf{N}_0)$, then*

$$\int_{-1}^1 P_n(x)\left\langle F(m), \frac{2m+1}{2}P_m(x)\right\rangle dx$$
$$= \left\langle F(m), \frac{2m+1}{2}\int_{-1}^1 P_m(x)P_n(x)\, dx\right\rangle. \tag{3.5}$$

Proof. The left-hand side in (3.5) can be written, by virtue of (3.1), as

$$\int_{-1}^1 P_n(x) f(x)\, dx, \tag{3.6}$$

where $f(x) = \{l'F(n)\}(x)$. This integral exists always since $f(x) \in C^\infty(-1, 1)$ in light of Proposition 3.1. Moreover, $f(x)$ is bounded on $[-1, 1]$

because there exist a constant $C > 0$ and $r \in \mathbf{N}_0$ such that, for every x, $-1 \le x \le 1$, one has [15, p. 19]

$$|f(x)| = \left| \left\langle F(n), \frac{2n+1}{2} P_n(x) \right\rangle \right| \le C \max_{0 \le k \le r} \lambda_{\alpha, k} \left\{ \frac{2n+1}{2} P_n(x) \right\}$$
$$= C \max_{0 \le k \le r} \sup_{n \in \mathbf{N}_0} \left\{ e^{-\alpha n} \frac{2n+1}{2} |x|^r |P_n(x)| \right\}$$
$$\le C \max_{0 \le k \le r} \sup_{n \in \mathbf{N}_0} \left\{ e^{-\alpha n} \frac{2n+1}{2} \right\} = C_1 < \infty.$$

On the other hand, by fixing arbitrarily $n \in \mathbf{N}_0$, the orthogonality relation for the Legendre polynomials

$$\frac{2m+1}{2} \int_{-1}^{1} P_m(x) P_n(x)\, dx = \delta_{m,n}$$

in other words, the Kronecker sequence $\delta_{m,n}$ considered as a function of m belongs obviously to $l(\mathbf{N}_0)$. Thus, the right-hand side in (3.5) is well defined too. To derive the equality, we could proceed using the technique of Riemann sums in a similar way as in [15, p. 148] with regard to the Hankel transform.

PROPOSITION 3.3. *For every $F(n) \in l'(\mathbf{N}_0)$ and $\Phi(l) \in l(\mathbf{N}_0)$ we have*

$$\langle \langle F(m), \delta_{m,n} \rangle, \Phi(n) \rangle = \langle F(m), \langle \delta_{m,n}, \Phi(n) \rangle \rangle.$$

Proof. We only outline the proof. Note, on the other hand, that $\langle F(m), \delta_{m,n} \rangle$ generates a regular member in $l'(\mathbf{N}_0)$, while $\langle \delta_{m,n}, \Phi(n) \rangle \in l(\mathbf{N}_0)$ since $\sum_{n=0}^{\infty} e^{\alpha n} |\delta_{m,n}| = e^{\alpha m}$ and, consequently, $\delta_{m,n}$ gives rise to a regular member in $l'(\mathbf{N}_0)$ as well. Therefore, the following steps are justified:

$$\langle F(m), \langle \delta_{m,n}, \Phi(n) \rangle \rangle = \left\langle F(m), \sum_{n=0}^{\infty} \delta_{m,n} \Phi(n) \right\rangle$$
$$= \sum_{n=0}^{\infty} \langle F(m), \delta_{m,n} \rangle \Phi(n)$$
$$= \langle \langle F(m), \delta_{m,n} \rangle, \Phi(n) \rangle.$$

We are at last ready to prove the paramount results of this section. Firstly, we establish the inversion formula of the discrete transform (1.7):

THEOREM 3.1. *Let $F(n)$ be an arbitrary generalized sequence in the space $l'(\mathbf{N}_0)$ and let $f(x)$ be the generalized discrete Legendre transform of $F(n)$ as defined by* (3.1). *Then,*

$$\left\{l'^{-1}f(x)\right\}(n) = F(n) = \int_{-1}^{1} P_n(x) f(x)\, dx \tag{3.7}$$

in the sense of equality in $l'(\mathbf{N}_0)$

Proof. Let any $\Phi(n) \in l(\mathbf{N}_0)$. By invoking Propositions 3.2 and 3.3 and definition (3.1) we may write

$$\begin{aligned}
&\left\langle \int_{-1}^{1} P_n(x) f(x)\, dx, \Phi(n) \right\rangle \\
&\quad = \left\langle \int_{-1}^{1} P_n(x) \left\langle F(m), \frac{2m+1}{2} P_m(x) \right\rangle dx, \Phi(n) \right\rangle \\
&\quad = \left\langle \left\langle F(m), \frac{2m+1}{2} \int_{-1}^{1} P_m(x) P_n(x)\, dx \right\rangle, \Phi(n) \right\rangle \\
&\quad = \langle\langle F(m), \delta_{m,n}\rangle, \Phi(n)\rangle = \langle F(m), \langle \delta_{m,n}, \Phi(n)\rangle\rangle .
\end{aligned} \tag{3.8}$$

So then,

$$\langle \delta_{m,n}, \Phi(n)\rangle = \sum_{n=0}^{\infty} \Phi(n)\, \delta_{m,n} = \Phi(m).$$

Hence, the right-hand side in (3.8) takes the value $\langle F(n), \Phi(n)\rangle$ and the proof is finished.

Next the uniqueness theorem is shown:

THEOREM 3.2. *Let $F(n)$ and $G(n)$ be arbitrary members in $l'(\mathbf{N}_0)$. If $f(x) = g(x)$, $x \in (-1, 1)$, where $f(x) = \{l'F(n)\}(x)$ and $g(x) = \{l'G(n)\}(x)$, then*

$$F(n) = G(n)$$

holds in the sense of equality in $l'(\mathbf{N}_0)$.

Proof. It is an immediate consequence of Theorem 3.1.

Finally, the main operational rule is derived:

THEOREM 3.3. *If $F(n) \in l'(\mathbf{N}_0)$, we get*

$$\left\{l'\left[\Lambda_n^k F(n)\right]\right\}(x) = x^k \{l'F(n)\}(x) \tag{3.9}$$

for each $k = 0, 1, 2, \ldots$.

Proof. By virtue of (3.1), (2.8), and (2.5) we easily obtain

$$\begin{aligned}\left\{l'\left[\Lambda_n^k F(n)\right]\right\}(x) &= \left\langle \Lambda_n^k F(n), \frac{2n+1}{2} P_n(x) \right\rangle \\ &= \left\langle F(n), \Lambda_n^{*k}\left(\frac{2n+1}{2} P_n(x)\right)\right\rangle \\ &= \left\langle F(n), x^k \frac{2n+1}{2} P_n(x) \right\rangle \\ &= x^k \left\langle F(n), \frac{2n+1}{2} P_n(x) \right\rangle = x^k \{l' F(n)\}(x).\end{aligned}$$

4. THE TRANSLATION OPERATOR AND THE CONVOLUTION

In accordance with I. I. Hirschman [5], we denote

$$\int_{-1}^{1} P_l(x) P_m(x) P_n(x)\, dx = \Pi(l, m, n). \tag{4.1}$$

From (1.2) and (1.1) it is formally deduced that

$$f(x) = \sum_{n=0}^{\infty} \frac{2n+1}{2} F(n) P_n(x) = \sum_{n=0}^{\infty} \frac{2n+1}{2} \left[\int_{-1}^{1} P_n(t) f(t)\, dt\right] P_n(x).$$

In particular, if we set $f(x) = P_l(x) P_m(x)$ $(l, m \in \mathbf{N}_0)$, the last equality becomes

$$P_l(x) P_m(x) = \sum_{n=0}^{\infty} \frac{2n+1}{2} \Pi(l, m, n) P_n(x) = \{l \Pi(l, m, n)\}(x), \tag{4.2}$$

which resembles the duplication formula [1, (4)]

$$P_n(x) P_n(y) = \frac{1}{\pi} \int_0^{\pi} P_n\left(xy + \sqrt{(1-x^2)(1-y^2)} \cos \alpha\right) d\alpha$$

in relation with the continuous Legendre transformation (1.1).

Expression (4.2) suggests the following definition

DEFINITION 4.1. Let $m \in \mathbf{N}_0$ be arbitrarily fixed. We define the translation t_m of the sequence $\Phi(l) \in l(\mathbf{N}_0)$ by means of

$$t_m \Phi(l) = \sum_{n=0}^{\infty} \frac{2n+1}{2} \Pi(l, m, n) \Phi(n). \tag{4.3}$$

Next we list some properties of the translation operator t_m:

(i) $t_m\Phi(l) = t_l\Phi(m)$, because of $\Pi(l,m,n) = \Pi(m,l,n)$.

(ii) Expression (4.2) can be rewritten

$$t_m P_l(x) = P_l(x) P_m(x).$$

(iii) $t_0\Phi(l) = \Phi(l)$, since

$$\Pi(l,0,n) = \begin{cases} 0, & \text{if } n \neq l \\ 2/(2l+1), & \text{if } n = l. \end{cases}$$

Note that $\Pi(l,m,n) = 0$ for $n > l + m$. This implies that the right-hand side in (4.3) reduces to a finite sum and, therefore, it has always a sense for any nonnegative pair of arbitrarily fixed integer numbers m and l.

However, we cannot guarantee that $t_m\Phi(l)$, with $m \in \mathbf{N}_0$ fixed, belongs to $l(\mathbf{N}_0)$ when $\Phi \in l(\mathbf{N}_0)$.

Next we introduce the space $\sigma(\mathbf{N}_0)$ which consists of all the complex-valued sequence $\{\psi(n)\}_{n\in\mathbf{N}_0}$ such that

$$\beta_{\gamma,k}(\psi(n)) = \sum_{n=0}^{\infty} e^{\gamma n}|\Lambda_n^{*k}\psi(n)| < \infty, \qquad k = 0,1,2,\ldots \tag{4.4}$$

with γ denoting a nonnegative parameter and the operator Λ_n^* being given by (2.9).

When endowed with the topology generated by the family of seminorms (4.4), $\sigma(\mathbf{N}_0)$ turns out to be a Fréchet space. We denote $\sigma'(\mathbf{N}_0)$ for the dual space of $\sigma(\mathbf{N}_0)$. It is worth emphasizing that $\sigma(\mathbf{N}_0) \subset l'(\mathbf{N}_0)$ in the sense that each member $\psi(n) \in \sigma(\mathbf{N}_0)$ gives rise to a regular member in $l'(\mathbf{N}_0)$ through (2.6). Indeed, any member $\psi(n) \in \sigma(\mathbf{N}_0)$ fulfills the condition (2.5) since

$$\sum_{n=0}^{\infty} e^{\alpha n}|\psi(n)| = \beta_{\alpha,0}(\psi) < \infty.$$

Conversely, a similar argument allows us to ensure that $l(\mathbf{N}_0) \subset \sigma'(\mathbf{N}_0)$, the inclusion being understood in the same sense commented above.

PROPOSITION 4.1. *Let $m \in \mathbf{N}_0$ be fixed. The operation*

$$\Phi(l) \mapsto \frac{2l+1}{2} t_m[\rho(l)\Phi(l)],$$

where $\rho(l)$ denotes a rational function whose denominator has no nonnegative root, is a continuous linear mapping from $\sigma(\mathbf{N}_0)$ into $l(\mathbf{N}_0)$.

Proof. The above mapping is clearly linear. To verify its continuity, we start from

$$\begin{aligned} e^{-\alpha l}\Lambda_l^{*k}\left\{\frac{2l+1}{2}t_m[\rho(l)\Phi(l)]\right\} &= e^{-\alpha l}\frac{2l+1}{2}\sum_{n=0}^{\infty}\left[\int_{-1}^{1}x^k P_l(x)P_m(x)P_n(x)\,dx\right] \\ &\quad\times\frac{2n+1}{2}\rho(n)\Phi(n), \end{aligned}$$

in view of (4.1) and (2.5). Thus, taking into account that $|P_j(x)| \leq 1$ for every $j \in \mathbf{N}_0$ and any $x \in [-1,1]$, we can find a constant $C_{\alpha,\gamma} > 0$ so that

$$\begin{aligned} \left|e^{-\alpha l}\Lambda_l^{*k}\left\{\frac{2l+1}{2}t_m[\rho(l)\Phi(l)]\right\}\right| &\leq (2l+1)e^{-\alpha l}\sum_{n=0}^{\infty}e^{-\gamma n}\frac{2n+1}{2}|\rho(n)|e^{\gamma n}|\Phi(n)| \\ &\leq C_{\alpha,\gamma}\sum_{n=0}^{\infty}e^{\gamma n}|\Phi(n)| \leq C_{\alpha,\gamma}\beta_{\gamma,0}(\Phi), \end{aligned}$$

whenever $\gamma > 0$. This means that

$$\gamma_{\alpha,k}\left\{\frac{2l+1}{2}t_m[\rho(l)\Phi(l)]\right\} \leq C_{\alpha,\gamma}\beta_{\gamma,0}(\Phi).$$

Our statement is implied by the last inequality and [15, p. 26].

The generalized translation operator t'_m acting on $F(n) \in l'(\mathbf{N}_0)$ is defined by means of

$$\langle t'_m F(l), \Phi(l)\rangle = \left\langle F(l), \frac{2l+1}{2}t_m\left[\frac{2}{2l+1}\Phi(l)\right]\right\rangle \tag{4.5}$$

for every $\Phi(l) \in \sigma(\mathbf{N}_0)$.

The next assertion is a simple consequence of definition (4.5), Proposition 4.1, and [15, Theorem 1.10-1].

THEOREM 4.1. *The generalized translation operator t'_m is a continuous linear mapping from $l'(\mathbf{N}_0)$ into $\sigma'(\mathbf{N}_0)$.*

The classical convolution for the discrete Legendre transformation was introduced by I. I. Hirschman [5, p. 339]. With the aid of the translation operator (4.3), it can be expressed as

$$\begin{aligned}(\Phi \circ \Psi)(n) &= \sum_{l=0}^{\infty} \frac{2l+1}{2} \Phi(l) t_n \Psi(l) \\ &= \sum_{l=0}^{\infty} \sum_{m=0}^{\infty} \frac{2l+1}{2} \frac{2m+1}{2} \Pi(l,m,n) \Phi(l) \Psi(m). \quad (4.6)\end{aligned}$$

PROPOSITION 4.2. *Assume $\Phi(n)$ and $\Psi(n)$ are testing sequences of the space $\sigma(\mathbf{N}_0)$. Then, $(\Phi \circ \Psi)(n) \in l(\mathbf{N}_0)$.*

Proof. In view of (4.6), it is easily seen that

$$\left| e^{-\alpha n} \Lambda_n^{*k} (\Phi \circ \Psi)(n) \right| \leq C_\alpha \beta_{\gamma,0}(\Phi) \beta_{\gamma,0}(\Psi),$$

C_α being a certain positive constant, in other words,

$$\lambda_{\alpha,k}\{(\Phi \circ \Psi)(n)\} \leq C_\alpha \beta_{\gamma,0}(\Phi) \beta_{\gamma,0}(\Psi)$$

for any $\gamma > 0$.

DEFINITION 4.2. The convolution of two members $F(n)$ and $G(n)$ in $l'(\mathbf{N}_0)$ will be defined by

$$\begin{aligned}&\langle F(n) \circ G(n), \Phi(n) \rangle \\ &\quad = \left\langle F(n), \frac{2n+1}{2} \left\langle t_n' G(m), \Phi(m) \right\rangle \right\rangle \\ &\quad \left\langle F(n), \frac{2n+1}{2} \left\langle G(m), \frac{2m+1}{2} t_n \left[\frac{2}{2m+1} \Phi(m) \right] \right\rangle \right\rangle, \quad (4.7)\end{aligned}$$

if it exists for every $\Phi \in l(\mathbf{N}_0)$.

In relation to this convolution we next proceed to establish the following assertion:

THEOREM 4.2. *If $F(n)$ and $G(n)$ are arbitrarily members in $l'(\mathbf{N}_0)$, then $(F \circ G)(n)$ exists and $(F \circ G)(n) \in \sigma'(\mathbf{N}_0)$.*

Proof. Let $\Phi(n)$ be any member in $\sigma(\mathbf{N}_0)$. By virtue of Proposition 4.1, $\langle G(m), ((2m+1)/2)t_n[(2/(2m+1))\Phi(m)]\rangle$ has a sense. All that remains to prove is that

$$\Phi^*(n) = \frac{2n+1}{2}\left\langle G(m), \frac{2m+1}{2}t_n\left[\frac{2}{2m+1}\Phi(m)\right]\right\rangle \tag{4.8}$$

is a sequence belonging to $l(\mathbf{N}_0)$.

Note, on the one hand, that the difference operator Λ_n commutes with the translation operator t_m:

$$\Lambda_l t_m \Phi(l) = t_m \Lambda_l \Phi(l). \tag{4.9}$$

On the other hand, by (i) and (4.9) we arrive immediately at

$$\begin{aligned} e^{-\alpha n}&\left|\Lambda_n^{*k}\frac{2n+1}{2}\left\langle G(m), \frac{2m+1}{2}t_m\left[\frac{2}{2n+1}\Phi(n)\right]\right\rangle\right| \\ &= e^{-\alpha n}\frac{2n+1}{2}\left|\left\langle G(m), \frac{2m+1}{2}t_n\left[\frac{2}{2m+1}\Lambda_m^{*k}\Phi(m)\right]\right\rangle\right|. \end{aligned} \tag{4.10}$$

Finally, there exist a positive constant C and a nonnegative integer r, depending on $G(n)$ but not on $\Phi(n)$ [15, Theorem 1.8-1], such that

$$\begin{aligned} &\left|\left\langle G(m), \frac{2m+1}{2}t_n\left[\frac{2}{2m+1}\Lambda_m^{*k}\Phi(m)\right]\right\rangle\right| \\ &\quad\le C\max_{0\le l\le r}\left[\lambda_{\alpha,l}\left\{\frac{2m+1}{2}t_n\left[\frac{2}{2m+1}\Lambda_m^{*k}\Phi(m)\right]\right\}\right] \\ &\quad= C\max_{0\le l\le r}\sup_{m\in\mathbf{N}_0}\left\{e^{-\alpha m}\frac{2m+1}{2}\left|t_n\left[\frac{2}{2m+1}(\Lambda_m^*)^{k+l}\Phi(m)\right]\right|\right\} \\ &\quad= C\max_{0\le l\le r}\sup_{m\in\mathbf{N}_0}\left\{e^{-\alpha m}\frac{2m+1}{2}\left|\sum_{s=0}^{\infty}\Pi(n,m,s)(\Lambda_s^*)^{k+l}\Phi(s)\right|\right\} \\ &\quad\le C\max_{0\le l\le r}\{\beta_{\alpha,k+l}(\Phi)\} < \infty, \end{aligned} \tag{4.11}$$

since $\Phi\in\sigma(\mathbf{N}_0)$. Inasmuch as $\Pi(n,m,s)\le 2$, no matter what the nonnegative integers m, n, and s may be, the constant C appearing in (4.11) does not depend on n and, consequently, from (4.8), (4.11), and (4.10) we conclude

$$\lambda_{\alpha,k}\{\phi^*(n)\} < \infty.$$

That is to say, $\Phi^*(n) \in l(\mathbf{N}_0)$ and $\langle F(n), \Phi^*(n)\rangle$ has a sense. Definitively, $(F \circ G)(n) \in \sigma'(\mathbf{N}_0)$.

COROLLARY 4.1. *If $F(n) \in l'(\mathbf{N}_0)$ and $G(n) \in \sigma(\mathbf{N}_0)$, then the convolution $H(n) = (F \circ G)(n)$ is given by*

$$H(n) = \left\langle F(m), \frac{2m+1}{2} t_n G(m) \right\rangle. \tag{4.12}$$

Proof. Recall that $\sigma(\mathbf{N}_0) \subset l'(\mathbf{N}_0)$. This fact implies that $H(n) = (F \circ G)(n)$ is well defined in $\sigma'(\mathbf{N}_0)$, in agreement with Theorem 4.2. On the other hand, since $((2m+1)/2)t_n G(m) \in l(\mathbf{N}_0)$ by invoking Proposition 4.1 with $\rho(l) = 1$, the right-hand side in (4.12) has a sense also.

Next, consider $\Phi \in \sigma(\mathbf{N}_0)$. Then,

$$\begin{aligned}
&\langle H(n), \Phi(n)\rangle \\
&\quad = \langle (F \circ G)(n), \Phi(n)\rangle \\
&\quad = \left\langle F(n), \frac{2n+1}{2}\left\langle G(m), \frac{2m+1}{2} t_n\left[\frac{2}{2m+1}\Phi(m)\right]\right\rangle\right\rangle \\
&\quad = \left\langle F(n), \frac{2n+1}{2}\sum_{m=0}^{\infty} G(m)\frac{2m+1}{2} t_n\left[\frac{2}{2m+1}\Phi(m)\right]\right\rangle
\end{aligned} \tag{4.13}$$

because $G(m) \in \sigma(\mathbf{N}_0)$ generates a regular member in $l'(\mathbf{N}_0)$ by means of (2.6).

Moreover, we may verify that

$$\begin{aligned}
&\left|\sum_{m=0}^{\infty}\frac{2m+1}{2}G(m)\sum_{l=0}^{\infty}\Pi(m,n,l)\Phi(l)\right| \\
&\qquad \le C_\alpha \sum_{m=0}^{\infty} e^{\alpha m}|G(m)| \sum_{l=0}^{\infty} e^{\alpha l}|\Phi(l)| = C_\alpha\, \beta_{\alpha,0}(G)\,\beta_{\alpha,0}(\Phi),
\end{aligned}$$

which assures that the order of summation can be changed. By this reason, making use of (4.3), we obtain

$$\sum_{m=0}^{\infty} G(m)\frac{2m+1}{2} t_n\left[\frac{2}{2m+1}\Phi(m)\right] = \sum_{l=0}^{\infty}\Phi(l) t_n G(l).$$

By substituting the last result in (4.13), we may write

$$
\begin{aligned}
\langle H(n), \Phi(n)\rangle &= \left\langle F(n), \frac{2n+1}{2}\sum_{l=0}^{\infty}\Phi(l)t_nG(l)\right\rangle \\
&= \sum_{l=0}^{\infty}\left\langle F(n), \frac{2n+1}{2}t_nG(l)\right\rangle\Phi(l) \\
&= \left\langle\left\langle F(n), \frac{2n+1}{2}t_nG(l)\right\rangle, \Phi(l)\right\rangle,
\end{aligned}
$$

seeing that we can establish that $\langle F(n), ((2n+1)/2)t_nG(l)\rangle \in l(\mathbf{N}_0)$ through a procedure quite analogous to that developed in Theorem 4.2 to verify that (4.8) belongs to $l(\mathbf{N}_0)$. This completes the proof.

COROLLARY 4.2. *If* $F(n), G(n) \in \sigma(\mathbf{N}_0)$ *then the classical convolution* (4.6) *is a special case of our definition* (4.7).

Proof. Inasmuch as $F(m) \in \sigma(\mathbf{N}_0)$, by invoking (4.12) and (2.6) one has

$$
\begin{aligned}
H(n) &= \sum_{m=0}^{\infty} F(m)\frac{2m+1}{2}t_nG(m) \\
&= \sum_{l=0}^{\infty}\sum_{m=0}^{\infty}\frac{2m+1}{2}\frac{2l+1}{2}\Pi(m,n,l)F(l)G(m),
\end{aligned}
$$

in accordance with the definition due to I. I. Hirschman [5, p. 339].

5. APPLICATIONS

This section is devoted to an application of the preceding theory to solve a partial difference equation. Thus, we propose to find a conventional solution $u(m,n)$ of the difference equation

$$
\frac{m+1}{2m+1}u(m+1,n) + \frac{m}{2m+1}u(m-1,n) - u(m,n+1) = 0 \quad (5.1)
$$

satisfying the condition

$$
u(m,0) = A(m) \in l'(\mathbf{N}_0). \quad (5.2)
$$

The above equation becomes

$$\Lambda_m u(m,n) - u(m,n+1) = 0, \tag{5.3}$$

where Λ_n denotes the operator (2.9).

By setting $\bar{u}(x,n) = l'\{u(m,n)\}$, applying the discrete Legendre transformation, and bearing in mind the operational rule (3.9), we convert (5.3) into the ordinary difference equation

$$x\bar{u}(x,n) + \bar{u}(x,n+1) = 0, \qquad -1 < x < 1,$$

whose general solution is

$$\bar{u}(x,n) = Cx^n.$$

Now, the condition (5.2) suggests that $C = a(x)$, with

$$a(x) = \bar{u}(x,0) = \{l'A(m)\}(x) = \left\langle A(m), \frac{2m+1}{2} P_m(x) \right\rangle.$$

Hence, the transform solution takes the form

$$\bar{u}(x,n) = a(x)x^n. \tag{5.4}$$

Finally, upon applying the inversion formula (Theorem 3.1) to (5.4), we obtain the solution

$$u(m,n) = \int_{-1}^{1} a(x) x^n P_m(x)\, dx. \tag{5.5}$$

Observe that the integral (5.5) always exists according to Proposition 3.1.

In certain particular cases we can deduce a more simple expression for the solution (5.5) that involves directly the initial condition (5.2). For this purpose, we need previously to prove the next result.

PROPOSITION 5.1. *If $F(n) \in l'(\mathbf{N}_0)$ and $G(n) \in l'(\mathbf{N}_0)$ are such that $F(n) \circ G(n) \in l'(\mathbf{N}_0)$ as well, then* $\{l'(F(n) \circ G(n))\} = \{l'F(n)\} \cdot \{l'G(n)\}$.

Proof. By virtue of definitions (3.1) and (4.7), and property (ii), we obtain

$$\begin{aligned}
\{l'(F(n) \circ G(n))\} &= \left\langle F(n) \circ G(n), \frac{2n+1}{2} P_n(x) \right\rangle \\
&= \left\langle F(n), \frac{2n+1}{2} \left\langle G(m), \frac{2m+1}{2} [t_n P_m(x)] \right\rangle \right\rangle \\
&= \left\langle F(n), \frac{2n+1}{2} P_n(x) \right\rangle \left\langle G(m), \frac{2m+1}{2} P_m(x) \right\rangle \\
&= \{l'F(n)\} \cdot \{l'G(n)\}.
\end{aligned}$$

Making use of this assertion and the inversion formula, we infer from (5.4) this other expression of our solution,

$$u(m,n) = A(m) \circ \Phi(m,n),$$

where [10, p. 421 (3)]

$$\Phi(m,n) = l'^{-1}\{x^n\}$$
$$= \begin{cases} 0, & \\ \quad \text{if } m+n = 1,3,5,\ldots \text{ or } m+n = 2,4,6,\ldots \\ \quad \text{and } m-n = 2,4,6,\ldots \\ \dfrac{\sqrt{\pi}\, n!}{2^n \Gamma((m+n+3)/2)\Gamma((n-m)/2+1)}, & \\ \quad \text{if } m+n = 0,2,4,\ldots;\ m-n \neq 2,4,6,\ldots. \end{cases}$$

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